Transverse and longitudinal dynamics of nonlinear intramolecular excitations on multileg ladder lattices

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A model of nonlinear intramolecular excitations on a multileg ladder lattice integrable by the inverse scattering transform is elaborated. The principal question of how to include the interchain linear coupling between excitations in the inverse scattering scheme is solved, and a detailed outline of the inverse scattering technique transforming the initially nonlinear problem into a linear one is given. The model permits a number of physically interesting applications related to striplike and bunchlike biological and condensed matter systems and in its partially continuous form to arrays of linearly and nonlinearly coupled optical fibers. The soliton dynamics across and along the chains for the cases of two-leg and three-leg ladder lattices is analyzed. The effect of an external magnetic field on the transverse dynamics of charged excitations on a three-leg ladder lattice with triangular cross section is studied and circular traveling as well as standing modes supporting the oscillating redistribution of soliton density between the chains are described. From the physical point of view it is reasonable to treat all transverse modes caused by interchain linear couplings as breathing modes, insofar as they correspond to the intrinsic degrees of freedom of a spatially constricted nonlinear wave packet moving uniformly as a whole along the chains.

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I. INTRODUCTION

Since the integrability of a continuous nonlinear Schrödinger equation in one spatial dimension was discovered [1], models of nonlinear Schrödinger type have played an exceptional role in physical applications for around three decades. They arise in rather different physical systems where the balance between dispersion and nonlinearity produces the fundamental entity known as a soliton. The applications of such models stretch from transport phenomena in low-dimensional biological [2–4] and condensed matter [4–7] systems to two-dimensional self-focusing [1,8] and one-dimensional self-modulation [1] of light in nonlinear media, to say nothing of light pulse propagation in optical fibers [9–11] and electric pulse propagation in nonlinear transmission lines [12]. The most intriguing ones are the multicomponent nonlinear models supporting nonlinear [8,9,11,13–15] or linear [16,17] couplings between their components, thereby prompting rather sophisticated effects of mode-mode interactions. However, as a rule only some of them are integrable, in particular, the well-known two-component Manakov model [8] admitting equal contributions from cross-phase and self-phase modulation effects as well as its discretized multicomponent versions that have recently appeared in the literature [18,19]. Though very useful for nonlinear optics, models of the Manakov type [8,18,19] are hardly suitable for the needs of biological and condensed matter systems, inasmuch as they serve as integrable models only for so-called incoherent solitons and do not describe the effects of linear (tunneling) couplings between the excitation amplitudes belonging to different chains. Meanwhile, real macromolecular systems should always be at least quasi-one-dimensional (i.e., should consist of several coupled chains), otherwise their lattice structure will be thermodynamically unstable [20]. Such reasoning has inevitably given rise to the development of discrete nonlinear models in more than one spatial dimension either to investigate the lattice dynamics of macromolecules themselves [21–23] or to study the propagation of nonlinear intramolecular excitations on them [16,17]. Recently, we have published two articles [24,25] dealing with nonlinear integrable models of intramolecular excitations on two-leg and multileg ladder lattices. The model on a two-leg ladder lattice [24] has been described in detail while for the model on a multileg ladder lattice [25] only a first rapid sketch has been given. Here we will try to fill in this gap by presenting the nonlinear model of intramolecular excitations on a multileg ladder lattice as broadly as now possible. Of course, results for the two-leg ladder model will always be developed from the multileg ladder model as the most general one. We also consider our integrable model as a plausible zero approximation at least for such known physically motivated models as arrays of tunnel-coupled nonlinear optical fibers [26–28] or models for the transport of excitation energy and charge in transversely coupled biological macromolecules [2–4]. In doing so we recall the powerful experience of other integrable models successfully applied as good starting positions in the analytical and numerical treatment of real physical systems [29–32]. It is worth noticing that apart from the interchain and intrachain linear couplings our model reproduces nonlinear cross-phase and self-phase modulation effects, too. In this respect we could regard it as some discrete multicomponent generalization of its continuous two-component counterparts [33–36].

The paper is organized as follows. In Sec. II we derive the nonlinear model of intramolecular excitations on a multileg ladder lattice with linear and nonlinear interchain couplings explicitly taken into account. We obtain the Lax representation of the model, thereby proving its exact integrability. In Sec. III we develop the basic principles of the inverse scattering transform as related to the model of interest and introduce the so-called modified transition matrix, playing a fundamental role in the integration of multicomponent nonlinear models with linear interchain couplings. In Sec. IV we derive Marchenko-type equations and in Sec. V we outline the general scheme of their solution and present explicitly the reduced soliton solution of the model. In Sec. VI we discuss the possible physical applications of the model and give a detailed analysis of soliton solutions as related to spatially constricted intramolecular excitations on two-leg and three-leg ladder lattices. Moreover, we describe the effects of a longitudinal magnetic field on the transverse soliton dynamics of charged excitations on a three-leg ladder lattice with triangular cross section and point out different types of transverse modes depending on the value of magnetic field.

Finally, in Sec. VII we summarize our results and discuss briefly the relationship of beating, standing, and traveling circular modes with breathing modes.

II. MODEL OF INTEREST AND ITS DERIVATION

Following the terminology of transport phenomena we define the quantities $q_\alpha(n)$ and $r_\alpha(n)$ to be the amplitudes of intramolecular excitation of a molecule sited on the αth chain within the nth unit cell. The longitudinal numerical coordinate n runs from minus to plus infinity, whereas the transverse one α runs from 1 to the number of chains, M. Furthermore, we consider the intrachain linear couplings between excitations to be of the nearest neighboring type, while the interchain ones are extended within each unit cell. As a result the whole structure of tunneling channels (bonds) will form some sophisticated multileg ladder with legs directed along the chains and rungs connecting all molecules within the same unit cell. Unfortunately, as usually happens in other discrete integrable models [18,19,37], we are not in a position to follow literally all of the nonlinear features of real physical systems, but the main part of them will be reproduced rather accurately. In any case the difference between the real and modeled nonlinearities can be reasonably taken into account even in describing the strongly localized states [29,30,38].

In order to derive the nonlinear model of intramolecular excitations on a multileg ladder lattice integrable by inverse scattering transform, we start with the standard Lax approach [39] and introduce two auxiliary linear problems,

$$\mathbf{u}(n+1|z)=L(n|z)\mathbf{u}(n|z), \tag{1}$$

$$d\mathbf{u}(n|z)/d\tau=A(n|z)\mathbf{u}(n|z), \tag{2}$$

on the $2M$-component column vector $\mathbf{u}(n|z)$ with the preassigned but appropriately chosen spectral operator $L(n|z)$ and the evolution operator $A(n|z)$ unknown *a priori*. Here τ stands for the dimensionless time. The spectral parameter z is assumed to be time independent as usual, while the explicit indication on the time dependences of other quantities will typically be omitted for the sake of brevity. According to the general rule [40] the spectral (1) and evolution (2) problems will be compatible, provided the so-called cross-differentiation condition

$$[d\mathbf{u}(m|z)/d\tau]_{m=n+1}=d\mathbf{u}(n+1|z)/d\tau \tag{3}$$

on the vector $\mathbf{u}(n|z)$ is imposed. As a result, the restriction

$$dL(n|z)/d\tau=A(n+1|z)L(n|z)-L(n|z)A(n|z) \tag{4}$$

on the operator $A(n|z)$ is obtainable. This equation allows us both to restore the explicit form of the evolution operator $A(n|z)$ and to isolate the nonlinear evolution model of interest, provided the sequence of powers in an expansion of $A(n|z)$ with respect to z is suitably selected. For our purposes the first nontrivial choice, when the power sequence in an expansion of $A(n|z)$ follows that of $L^2(n|z)$, turns out to be rather satisfactory.

In accordance with our aspirations the most suitable matrix form of the spectral operator $L(n|z)$ is postulated as follows:

$$L(n|z)=\begin{pmatrix} zI & F(n)E \\ EG(n) & z^{-1}I \end{pmatrix}. \tag{5}$$

Hereafter the quantities I, E, T, and $F(n), G(n)$ stand for $M\times M$ submatrices defined by the expressions

$$I\equiv[I_{\alpha\beta}]=[\delta_{\alpha\beta}]. \tag{6}$$

$$E\equiv[E_{\alpha\beta}]=[1], \tag{7}$$

$$T\equiv[t_{\alpha\beta}], \tag{8}$$

$$F(n)\equiv[F_{\alpha\beta}(n)]=[iq_\alpha(n)\delta_{\alpha\beta}]/\sqrt{M}, \tag{9}$$

$$G(n)\equiv[G_{\alpha\beta}(n)]=[i\delta_{\alpha\beta}r_\beta(n)]/\sqrt{M}. \tag{10}$$

Then, the above mentioned guiding scheme allow us on the one hand to fix the form of the evolution operator explicitly,

$$A(n|z)=\begin{pmatrix} iz^2I-iF(n)EEG(n-1)+iT & izF(n)E-iz^{-1}F(n-1)E \\ izEG(n-1)-iz^{-1}EG(n) & -iz^{-2}I+iEG(n)F(n-1)E \end{pmatrix}, \tag{11}$$

and on the other hand to obtain the exactly integrable model of some intramolecular nonlinear excitations on a multileg (M-leg) ladder lattice,

$$i\dot{q}_\alpha(n)+\sum_{\beta=1}^{M} t_{\alpha\beta}q_\beta(n)+[q_\alpha(n+1)+q_\alpha(n-1)]\times\left(1+\sum_{\beta=1}^{M} q_\beta(n)r_\beta(n)\right)=\sum_{\beta=1}^{M}[q_\alpha(n-1)q_\beta(n)-q_\alpha(n)q_\beta(n-1)]r_\beta(n), \tag{12}$$

$$-i\dot{r}_\alpha(n)+\sum_{\beta=1}^{M} r_\beta(n)t_{\beta\alpha}+[r_\alpha(n+1)+r_\alpha(n-1)]\times\left(1+\sum_{\beta=1}^{M} r_\beta(n)q_\beta(n)\right)=\sum_{\beta=1}^{M}[r_\alpha(n+1)r_\beta(n)-r_\alpha(n)r_\beta(n+1)]q_\beta(n), \tag{13}$$

where the overdot stands for the derivative with respect to the dimensionless time τ. As we have already pointed out, the longitudinal numerical coordinate n runs from minus to plus infinity, while the transverse one α runs from 1 to the number of chains (legs), M. This statement will always be applied whenever the quantities n and α are mentioned. From Eqs. (12) and (13) it is readily seen that the parameters $t_{\alpha\beta}$ are responsible for the interchain linear coupling, while the constants of the intrachain linear coupling are normalized to unity. The coupling parameters $t_{\alpha\beta}$ are supposed to be arbitrary and even time dependent for the time being.

By the way, when starting with the expressions (5) and (11) for $L(n|z)$ and $A(n|z)$ as known we can treat the matrix equation (4) as the Lax or zero-curvature representation of our model (12) and (13). The existence of such a representation is actually a basic condition sufficient to handle the exact integrability of the model under investigation.

A comprehensive analysis carried out by inverse scattering transform shows that one-soliton amplitudes cancel the right-hand terms in Eqs. (12) and (13) identically, and convert the initial multileg model (12) and (13) into the simpler one,

$$i\dot{q}_\alpha(n)+\sum_{\beta=1}^{M} t_{\alpha\beta}q_\beta(n)+[q_\alpha(n+1)+q_\alpha(n-1)]\times\left(1+\sum_{\beta=1}^{M} q_\beta(n)r_\beta(n)\right)=0, \tag{14}$$

$$-i\dot{r}_\alpha(n)+\sum_{\beta=1}^{M} r_\beta(n)t_{\beta\alpha}+[r_\alpha(n+1)+r_\alpha(n-1)]\times\left(1+\sum_{\beta=1}^{M} r_\beta(n)q_\beta(n)\right)=0. \tag{15}$$

Provided the matrix $[t_{\alpha\beta}]$ is Hermitian, $t^*_{\alpha\beta}=t_{\beta\alpha}$, this last model (14) and (15) gains a direct physical implication, inasmuch as then its amplitudes can be linked by one of the reductions $r_\alpha(n)=q^*_\alpha(n)$ or $r_\alpha(n)=-q^*_\alpha(n)$. Indeed, as long as the initial [Eqs. (12) and (13)] and simplified [Eqs. (14) and (15)] models conserve the quantity

$$\sum_{m=-\infty}^{\infty}\ln\left[1+\sum_{\beta=1}^{M} q_\beta(m)r_\beta(m)\right],$$

we can introduce the corrected amplitudes

$$Q_\alpha(n)=q_\alpha(n)\sqrt{\frac{\pm\ln\left[1+\sum_{\beta=1}^{M} q_\beta(n)r_\beta(n)\right]}{\sum_{\beta=1}^{M} q_\beta(n)r_\beta(n)}}, \tag{16}$$

$$R_\alpha(n)=r_\alpha(n)\sqrt{\frac{\pm\ln\left[1+\sum_{\beta=1}^{M} q_\beta(n)r_\beta(n)\right]}{\sum_{\beta=1}^{M} q_\beta(n)r_\beta(n)}}, \tag{17}$$

revealing at $r_\alpha(n)=\pm q^*_\alpha(n)$ all the necessary features of probability amplitudes. Of course, both models can easily be reformulated in terms of $Q_\alpha(n)$ and $R_\alpha(n)$.

III. INVERSE SCATTERING SCHEME: SOME BASIC DEFINITIONS AND USEFUL PREPARATORY RESULTS

Although we already know that the multicomponent model (12) and (13) is exactly integrable, nevertheless the general method of its integration requires a number of essential modifications as compared with purely one-chain models [37,40,41] or multichain models without interchain linear coupling [18,19]. Thus, we should include the effects of interchain linear coupling in the inverse scattering scheme itself, which will be shown to be a rather nontrivial move. Fortunately, experience acquired by integrating the nonlinear model on a two-leg ladder lattice [24] will allow us to overcome the difficulties.

In what follows we restrict ourselves to the case of potentials $q_\alpha(n)$ and $r_\alpha(n)$ rapidly decreasing at infinity $|n|\to\infty$ and define the left $\{\boldsymbol{\varphi}_j(n|z)\}$ and right $\{\boldsymbol{\psi}_j(n|z)\}$ Jost bases ($j=1,2,\ldots,2M$) as the vector sets satisfying the auxiliary spectral problem (1) and (5) and fixed by the asymptotic conditions

$$\varphi_{ij}(n|z)\sim\delta_{ij}\left(\sum_{k=1}^{M}\delta_{jk}z^n+\sum_{k=M+1}^{2M}\delta_{jk}z^{-n}\right)\quad\text{as}\quad n\to-\infty, \tag{18}$$

$$\psi_{ij}(n|z)\sim\delta_{ij}\left(\sum_{k=1}^{M}\delta_{jk}z^n+\sum_{k=M+1}^{2M}\delta_{jk}z^{-n}\right)\quad\text{as}\quad n\to+\infty. \tag{19}$$

Here $\varphi_{ij}(n|z)$ and $\psi_{ij}(n|z)$ are the ith components of vectors $\boldsymbol{\varphi}_j(n|z)$ and $\boldsymbol{\psi}_j(n|z)$, respectively. Then, the transition ma trix $[a_{jk}(z)]$ gives the transformation from one basis to the other,

$$\boldsymbol{\varphi}_k(n|z)=\sum_{j=1}^{2M}\boldsymbol{\psi}_j(n|z)a_{jk}(z)\quad(k=1,2,\ldots,2M). \tag{20}$$

Conversely, supposing that the Jost bases are known, the relation

$$a_{ij}(z)=\frac{\underset{k=1}{\overset{2M}{\mathrm{W}}}\{(1-\delta_{ik})\boldsymbol{\psi}_k(n|z)+\delta_{ik}\boldsymbol{\varphi}_j(n|z)\}}{\underset{k=1}{\overset{2M}{\mathrm{W}}}\{\boldsymbol{\psi}_k(n|z)\}} \tag{21}$$

for the matrix elements $a_{ij}(z)$ can easily be obtained. Here $\mathrm{W}_{k=1}^{2M}\{\mathbf{v}_k(n|z)\}$ stands for the Wronskian of any $2M$ solutions $\mathbf{v}_1(n|z),\mathbf{v}_2(n|z),\ldots,\mathbf{v}_{2M}(n|z)$ of the spectral problem (1),(5) and it is defined by the identity

$$\underset{k=1}{\overset{2M}{\mathrm{W}}}\{\mathbf{v}_k(n|z)\}\equiv\det[v_{ik}(n|z)] \tag{22}$$

with $v_{ik}(n|z)$ denoting the ith component of the vector $\mathbf{v}_k(n|z)$.

Taking the Wronskian from both parts of the transforming equations (20) we come to the normalizing condition

$$\det[a_{ij}(z)]=\frac{\underset{k=1}{\overset{2M}{\mathrm{W}}}\{\boldsymbol{\varphi}_k(n|z)\}}{\underset{k=1}{\overset{2M}{\mathrm{W}}}\{\boldsymbol{\psi}_k(n|z)\}}=\prod_{m=-\infty}^{\infty}\left(1+\sum_{\beta=1}^{M}q_\beta(m)r_\beta(m)\right), \tag{23}$$

where in the last step the following relations have been used:

$$\underset{k=1}{\overset{2M}{\mathrm{W}}}\{\boldsymbol{\varphi}_k(n|z)\}=\prod_{m=-\infty}^{n-1}\left(1+\sum_{\beta=1}^{M}q_\beta(m)r_\beta(m)\right), \tag{24}$$

$$\underset{k=1}{\overset{2M}{\mathrm{W}}}\{\boldsymbol{\psi}_k(n|z)\}=\prod_{m=n}^{\infty}\left(1+\sum_{\beta=1}^{M}q_\beta(m)r_\beta(m)\right)^{-1}, \tag{25}$$

based upon the combination of the spectral problem (1),(5) and the asymptotic conditions (18),(19).

Finally, it can be shown that two sets of vectors

$$\{\boldsymbol{\varphi}_1(n|z)z^{-n},\ldots,\boldsymbol{\varphi}_M(n|z)z^{-n},\ \boldsymbol{\psi}_{M+1}(n|z)z^{n},\ldots,\boldsymbol{\psi}_{2M}(n|z)z^{n}\} \tag{26}$$

and

$$\{\boldsymbol{\psi}_1(n|z)z^{-n},\ldots,\boldsymbol{\psi}_M(n|z)z^{-n},\ \boldsymbol{\varphi}_{M+1}(n|z)z^{n},\ldots,\boldsymbol{\varphi}_{2M}(n|z)z^{n}\} \tag{27}$$

are analytic outside ($|z|>1$) and inside ($|z|<1$) the unit circle, respectively, provided the potentials $q_\alpha(n)$ and $r_\alpha(n)$ decrease sufficiently rapidly as $|n|\to\infty$. These properties enable us to seek the vectors of the right Jost basis in the form

$$\boldsymbol{\psi}_j(n|z)=\sum_{l=n}^{\infty}\mathbf{K}_j(n|l)\left(\sum_{k=1}^{M}\delta_{jk}z^{l}+\sum_{k=M+1}^{2M}\delta_{jk}z^{-l}\right),\qquad j=1,2,3,\ldots,M. \tag{28}$$

After substitution into the spectral equation (1) with $L(n|z)$ given by Eq. (5) the expansions (28) yield the relationships between the amplitudes $q_\alpha(n)$ and $r_\alpha(n)$ and the components $K_{ij}(n|m)$ of column vector $\mathbf{K}_j(n|m)$ as follows:

$$F(n)E=-\overset{-+}{K}(n|n+1)[\overset{++}{K}(n|n)]^{-1}, \tag{29}$$

$$EG(n)=-\overset{+-}{K}(n|n+1)[\overset{--}{K}(n|n)]^{-1}. \tag{30}$$

Here $\overset{--}{K}(n|m)$, $\overset{++}{K}(n|m)$ and $\overset{-+}{K}(n|m)$, $\overset{+-}{K}(n|m)$ are the diagonal and off-diagonal $M\times M$ submatrices of the $2M\times 2M$ matrix $[K_{ij}(n|m)]$, respectively, or more precisely

$$[K_{ij}(n|m)]\equiv\begin{pmatrix}\overset{--}{K}(n|m) & \overset{-+}{K}(n|m)\\ \overset{+-}{K}(n|m) & \overset{++}{K}(n|m)\end{pmatrix}. \tag{31}$$

For the sake of brevity, we will adopt a similar blocklike representation for any $2M\times 2M$ matrix whenever it is needed.

To integrate the nonlinear model (12) and (13) we should obtain a set of equations for the column vectors $\mathbf{K}_j(n|m)$ (equations of Marchenko type [42]) and then try to solve them with respect to some particular matrix elements $K_{ij}(n|m)$ involved in the relations (29) and (30). At this point, in contrast to the one-chain models [37,40,41] and the multichain models without interchain linear coupling [18,19], where the inverse scattering scheme is based on the analytical properties of Jost vectors side by side with the analytical properties of diagonal submatrices of the transition matrix, we inevitably arrive at a substantially more general way of reasoning. Indeed, although in the case of the multichain model (12) and (13) the analytical properties of the Jost vectors are still detectable, the analytical properties of the transition coefficients $a_{jk}(z)$ cannot be revealed *a priori* and, furthermore, are proven to be quite unnecessary. Instead, the entire logic of the problem leads us to the modified transition matrix $[\alpha_{jk}(z)]$ given by the combinations

$$\alpha_{jk}(z)=\delta_{jk}\det\overset{--}{a}(z)\qquad(\text{at }j=1,2,\ldots,M;\quad k=1,2,\ldots,M), \tag{32}$$

$$\alpha_{jk}(z)=\sum_{i=1}^{M} a_{ji}(z)\partial[\det \overset{--}{a}(z)]/\partial a_{ki}(z)$$

$$(\text{at } j=M+1,M+2,\ldots,2M;\quad k=1,2,\ldots,M), \quad (33)$$

$$\alpha_{jk}(z)=\sum_{i=M+1}^{2M} a_{ji}(z)\partial[\det \overset{++}{a}(z)]/\partial a_{ki}(z)$$

$$(\text{at } j=1,2,\ldots,M;\quad k=M+1,M+2,\ldots,2M), \quad (34)$$

$$\alpha_{jk}(z)=\delta_{jk}\det \overset{++}{a}(z)$$

$$(\text{at } j=M+1,M+2,\ldots,2M;\quad k=M+1,M+2,\ldots,2M). \quad (35)$$

Here $\overset{--}{a}(z)$ and $\overset{++}{a}(z)$ are the diagonal $M\times M$ submatrices of the $2M\times 2M$ matrix $[a_{ij}(z)]$.

In what follows only the analytical properties of the diagonal elements $\alpha_{kk}(z)$ of the modified transition matrix are required. Fortunately, they are precisely those elements admitting a thorough treatment. Thus, the expressions

$$\alpha_{11}(z)=\frac{\underset{k=1}{\overset{2M}{\mathrm{W}}}\left\{\sum_{j=1}^{M}\delta_{jk}\boldsymbol{\varphi}_k(n|z)+\sum_{j=M+1}^{2M}\delta_{jk}\boldsymbol{\psi}_k(n|z)\right\}}{\underset{k=1}{\overset{2M}{\mathrm{W}}}\{\boldsymbol{\psi}_k(n|z)\}}, \quad (36)$$

$$\alpha_{2M2M}(z)=\frac{\underset{k=1}{\overset{2M}{\mathrm{W}}}\left\{\sum_{j=1}^{M}\delta_{jk}\boldsymbol{\psi}_k(n|z)+\sum_{j=M+1}^{2M}\delta_{jk}\boldsymbol{\varphi}_k(n|z)\right\}}{\underset{k=1}{\overset{2M}{\mathrm{W}}}\{\boldsymbol{\psi}_k(n|z)\}}, \quad (37)$$

taken at $n\to\infty$ show that $\alpha_{11}(z)$ and $\alpha_{2M2M}(z)$ are analytic outside ($|z|>1$) and inside ($|z|<1$) the unit circle, respectively.

We complete this section by presenting the evolution equations for the elements of the modified transition matrix,

$$\dot{\alpha}_{jk}(z)=i\sum_{j'=1}^{M}\delta_{j'j}\left((z+z^{-2})\alpha_{jk}(z)+\sum_{i=1}^{M}t_{ji}\alpha_{ik}(z)\right)$$
$$\times\sum_{k'=M+1}^{2M}\delta_{kk'}-i\sum_{j'=M+1}^{2M}\delta_{j'j}$$
$$\times\left((z+z^{-2})\alpha_{jk}(z)+\sum_{i=1}^{M}\alpha_{ji}(z)t_{ik}\right)\sum_{k'=1}^{M}\delta_{kk'}$$

$$(j=1,2,\ldots,2M;\quad k=1,2,3,\ldots,2M). \quad (38)$$

These equations have been derived based on the standard observation that at every $j=1,2,\ldots,2M$ the combination $\dot{\boldsymbol{\varphi}}_j(n|z)-A(n|z)\boldsymbol{\varphi}_j(n|z)$ satisfies the spectral problem (1),(5) and, consequently, it is presentable by some linear superposition of the left Jost vectors.

IV. MARCHENKO EQUATIONS

To proceed further with the fundamental aspects of the inverse scattering scheme it is helpful to rearrange the interbasis link (20) into the form

$$\mathbf{S}_k(n|z)\alpha_{kk}(z)=\sum_{j=1}^{2M}\boldsymbol{\psi}_j(n|z)\alpha_{jk}(z),\quad k=1,2,\ldots,M, \quad (39)$$

with the scattering vectors $\mathbf{S}_k(n|z)$ introduced in the following way:

$$\mathbf{S}_k(n|z)\alpha_{kk}(z)\equiv\sum_{i=1}^{M}\boldsymbol{\varphi}_i(n|z)\sum_{j=1}^{M}\delta_{jk}\partial[\det\overset{--}{a}(z)]/\partial a_{ki}(z)$$
$$+\sum_{i=M+1}^{2M}\boldsymbol{\varphi}_i(n|z)\sum_{j=M+1}^{2M}\delta_{jk}$$
$$\times\partial[\det\overset{++}{a}(z)]/\partial a_{ki}(z). \quad (40)$$

An appropriate analysis of scattering vectors $\mathbf{S}_k(n|z)$, similar to that described by Toda [41], yields the limiting formulas

$$\sum_{i=1}^{M}\delta_{ik}\lim_{|z|\to\infty}\mathbf{S}_k(n|z)z^{-n}+\sum_{i=M+1}^{2M}\delta_{ik}\lim_{|z|\to 0}\mathbf{S}_k(n|z)z^{n}=\mathbf{J}_k,$$
$$k=1,2,\ldots,2M, \quad (41)$$

and

$$\lim_{|z|\to\infty}a_{jk}(z)=\delta_{jk},\quad j,k=1,2,\ldots,M, \quad (42)$$

$$\lim_{|z|\to 0}a_{jk}(z)=\delta_{jk},\quad j,k=M+1,M+2,\ldots,2M, \quad (43)$$

relevant for future contour integration and reconstruction of diagonal matrix elements $\alpha_{kk}(z)$, respectively. Here $\mathbf{J}_k$ is the column vector with the ith component equal to δ_{ik}.

Assuming $\alpha_{kk}(z)$ at $|z|=1$ to be nonzero, we operate on the rearranged interbasis relation (39) with

$$\frac{1}{2\pi i}\oint\frac{dz}{\alpha_{kk}(z)}\left[\sum_{i=1}^{M}\delta_{ik}z^{-m-1}+\sum_{i=M+1}^{2M}\delta_{ik}z^{m-1}\right]\ldots \quad (44)$$

to find the set of equations

$$\mathbf{K}_k(n|m)+\sum_{l=n}^{\infty}\sum_{j=1}^{2M}\mathbf{K}_j(n|l)F_{jk}(l+m)=\mathbf{J}_k\delta_{nm}$$

$$(m\geqslant n,\ \ k=1,2,\ldots,2M) \quad (45)$$

of Marchenko type [42]. Here the matrix elements $F_{jk}(n)$ of the kernel operator are given by the expressions

$$F_{jk}(n)=\frac{1}{2\pi i}\oint_{|z|=1}dz\, z^{-n-1}\frac{\alpha_{jk}(z)}{\alpha_{kk}(z)}+\sum_{r=1}^{N_{\rm ext}} z_{rk}^{-n-1}\frac{\alpha_{jk}(z_{rk})}{\alpha'_{kk}(z_{rk})}$$

$$\text{at } j=M+1,M+2,\ldots,2M;\quad k=1,2,\ldots,M, \tag{46}$$

$$F_{jk}(n)=\frac{1}{2\pi i}\oint_{|z|=1}dz\, z^{n-1}\frac{\alpha_{jk}(z)}{\alpha_{kk}(z)}-\sum_{r=1}^{N_{\rm int}} z_{rk}^{n-1}\frac{\alpha_{jk}(z_{rk})}{\alpha'_{kk}(z_{rk})}$$

$$\text{at } j=1,2,\ldots,M;\quad k=M+1,M+2,\ldots,2M, \tag{47}$$

$$F_{jk}(n)\equiv 0 \quad \text{otherwise}, \tag{48}$$

where z_{rk} stands for the rth root of the equation $\alpha_{kk}(z)=0$, $\alpha'_{kk}(z_{rk})$ refers to the derivative $[d\alpha_{kk}(z)/dz]_{z=z_{rk}}$, and $N_{\rm ext}$ and $N_{\rm int}$ mark the total number of roots of the equations $\alpha_{11}(z)=0$ and $\alpha_{2M2M}(z)=0$, respectively. Here we would like to stress that although the equalities (46)–(48) have been found with the understanding of simple roots z_{rk}, the case of multiple roots can evidently be covered by obtaining limiting expressions on the final results.

The dependences of the scattering data $\alpha_{jk}(z)/\alpha_{kk}(z)$ as well as $\alpha_{jk}(z_{rk})/\alpha'_{kk}(z_{rk})$ and z_{rk} on time determine the dynamics of nonlinear intramolecular excitations on a multileg ladder lattice via solutions of the Marchenko equations. However, in contrast to the one-chain models and multichain models without interchain linear couplings, this dynamics turns out to be more complicated and even in the one-soliton case it reproduces both the traditional translational motion of excitation density along the chains as well as its transverse temporal redistribution between the chains. In terms of the scattering data, the effect of transverse redistribution arises from linear mixing between the elements of the modified scattering matrix caused by the interchain linear coupling $t_{\alpha\beta}$ $(\alpha\neq\beta)$. In principle, we can always obtain the time dependences of the scattering data explicitly, by previously integrating the evolution equations (38) for the elements of the modified scattering matrix at any particular choice of coupling parameters $t_{\alpha\beta}$ $(\alpha,\beta=1,2,\ldots,M)$.

V. SOLITON SOLUTIONS

For practical purposes it is worthwhile to separate the equations for $\mathbf{K}_j(n|m)$ with $j=1,2,\ldots,M$ from those for $\mathbf{K}_j(n|m)$ with $j=M+1,M+2,\ldots,2M$ and to reshape the Marchenko equations (45) into the form

$$\mathbf{K}_i(n|m)-\sum_{l=n}^{\infty}\sum_{p=n}^{\infty}\sum_{j=1}^{M}\sum_{k=M+1}^{2M}\mathbf{K}_j(n|l)F_{jk}(l+p)F_{ki}(p+m)$$
$$=\mathbf{J}_i\delta_{nm}-\sum_{k=M+1}^{2M}\mathbf{J}_kF_{ki}(n+m)$$
$$(m\geqslant n;\;\; i=1,2,\ldots,M), \tag{49}$$

$$\mathbf{K}_i(n|m)-\sum_{l=n}^{\infty}\sum_{p=n}^{\infty}\sum_{j=M+1}^{2M}\sum_{k=1}^{M}\mathbf{K}_j(n|l)F_{jk}(l+p)F_{ki}(p+m)$$
$$=\mathbf{J}_i\delta_{nm}-\sum_{k=1}^{M}\mathbf{J}_kF_{ki}(n+m)$$
$$(m\geqslant n;\;\; i=M+1,M+2,\ldots,2M). \tag{50}$$

To adapt these equations for the needs of multisolitonic solutions we must equalize the scattering data of the continuous spectrum $\alpha_{jk}(z)/\alpha_{kk}(z)$ to zero, either on and inside $(|z|\leqslant 1)$ or on and outside $(|z|\geqslant 1)$ the unit circle, depending on what combinations of indices $(j=M+1,M+2,\ldots,2M;\; k=1,2,\ldots,M)$ or $(j=1,2,\ldots,M;\; k=M+1,M+2,\ldots,2M)$ are chosen (the so-called unreflectional case). Then, on the one hand, the matrix elements $F_{jk}(l+m)$ of the kernel operator become degenerate [see Eqs. (46)–(48)] and, on the other, the form of the diagonal matrix elements $\alpha_{kk}(z)$ can be reconstructed explicitly:

$$\alpha_{kk}(z)=\prod_{s=1}^{N}\frac{z^2-\exp(\mu_s+ip_s)}{z^2-\exp(-\nu_s+iq_s)},\quad k=1,2,\ldots,M, \tag{51}$$

$$\alpha_{kk}(z)=\prod_{s=1}^{N}\frac{z^{-2}-\exp(\nu_s-iq_s)}{z^{-2}-\exp(-\mu_s-ip_s)},$$
$$k=M+1,M+2,\ldots,2M. \tag{52}$$

Here p_s and q_s are real constants, whereas μ_s and ν_s are positive real constants. Except for the restrictions imposed by the assumed simplicity of roots z_{rk}, the constants p_s, q_s, μ_s, ν_s are supposed to be arbitrary in all other respects. Finally, N represents an arbitrary but fixed positive integer, being the number of solitons in some particular multisoliton solution. Evidently $N_{\rm ext}=N_{\rm int}=2N$.

Despite being valid only for the unreflectional case, Eqs. (51) and (52) are consistent with the analyticity conditions $[\alpha_{11}(z)$ is analytical at $|z|>1$ and $\alpha_{2M2M}(z)$ is analytical at $|z|<1]$ and the limiting conditions $[\lim_{|z|\to\infty}\alpha_{11}(z)=1$ and $\lim_{|z|\to 0}\alpha_{2M2M}(z)=1]$, as well as with the normalizing condition (23) and the parity conditions $\alpha_{kk}(-z)=\alpha_{kk}(z)$. Though not mentioned earlier, the conditions $\alpha_{kk}(-z)=\alpha_{kk}(z)$ can easily be proved, at least for rapidly decreasing potentials $q_\alpha(n)$ and $r_\alpha(n)$ close to those on the compact support. We observe, by the way, that all other nonzero elements of the modified transition matrix happen to be odd functions of the spectral parameter $\alpha_{jk}(-z)=-\alpha_{jk}(z)$ $(j=M+1,M+2,\ldots,2M;\; k=1,2,\ldots,M$ and $j=1,2,\ldots,M;\; k=M+1,M+2,\ldots,2M)$.

Manipulating the Marchenko equations (49) and (50) in a way standard for integral equations with degenerate kernels [43] and using all the parity conditions of the modified transition matrix just mentioned, we find

$$\mathbf{K}_i(n|m)=\sum_{s'=1}^{N}\sum_{j=M+1}^{2M}\mathbf{X}_j^{s'q}(n)\sin^2\left(\frac{\pi m}{2}\right)\sum_{s''=1}^{N}\sum_{k=1}^{M}\exp(-\eta_{s''r}m)C_{s'qs''r}(n)b_{jk}^{s'q}b_{ki}^{s''r}$$
$$+\sum_{s'=1}^{N}\sum_{j=M+1}^{2M}\mathbf{Y}_j^{s'q}(n)\cos^2\left(\frac{\pi m}{2}\right)\sum_{s''=1}^{N}\sum_{k=1}^{M}\exp(-\eta_{s''r}m)S_{s'qs''r}(n)b_{jk}^{s'q}b_{ki}^{s''r}+\mathbf{J}_i\delta_{nm}-\left[\sin^2\left(\frac{\pi n}{2}\right)\cos^2\left(\frac{\pi m}{2}\right)\right.$$
$$\left.+\cos^2\left(\frac{\pi n}{2}\right)\sin^2\left(\frac{\pi m}{2}\right)\right]\sum_{s'=1}^{N}\exp[-\eta_{s'r}(n+m)]\sum_{k=1}^{M}\mathbf{J}_k b_{ki}^{s'r}\quad(m\geqslant n;\ \ i=M+1,M+2,\ldots,2M),\tag{53}$$

$$\mathbf{K}_i(n|m)=\sum_{s'=1}^{N}\sum_{j=1}^{M}\mathbf{X}_j^{s'r}(n)\sin^2\left(\frac{\pi m}{2}\right)\sum_{s''=1}^{N}\sum_{k=M+1}^{2M}\exp(-\eta_{s''q}m)C_{s'rs''q}(n)b_{jk}^{s'r}b_{ki}^{s''q}$$
$$+\sum_{s'=1}^{N}\sum_{j=1}^{M}\mathbf{Y}_j^{s'r}(n)\cos^2\left(\frac{\pi m}{2}\right)\sum_{s''=1}^{N}\sum_{k=M+1}^{2M}\exp(-\eta_{s''q}m)S_{s'rs''q}(n)b_{jk}^{s'r}b_{ki}^{s''q}+\mathbf{J}_i\delta_{nm}-\left[\sin^2\left(\frac{\pi n}{2}\right)\cos^2\left(\frac{\pi m}{2}\right)\right.$$
$$\left.+\cos^2\left(\frac{\pi n}{2}\right)\sin^2\left(\frac{\pi m}{2}\right)\right]\sum_{s'=1}^{N}\exp[-\eta_{s'q}(n+m)]\sum_{k=M+1}^{2M}\mathbf{J}_k b_{ki}^{s'q}\quad(m\geqslant n;\ \ i=1,2,\ldots,M),\tag{54}$$

where the $2M$-component column vectors $\mathbf{X}_i^{sq}(n),\mathbf{Y}_i^{sq}(n)$ with $i=M+1,M+2,\ldots,2M$ and $\mathbf{X}_i^{sr}(n),\mathbf{Y}_i^{sr}(n)$ with $i=1,2,\ldots,M$ are determined from the following four sets of linear algebraic equations:

$$\mathbf{X}_i^{sq}(n)-\sum_{s'=1}^{N}\sum_{j=M+1}^{2M}\mathbf{X}_j^{s'q}(n)\sum_{s''=1}^{N}\sum_{k=1}^{M}C_{s'qs''r}(n)S_{s''rsq}(n)b_{jk}^{s'q}b_{ki}^{s''r}$$
$$=\sin^2\left(\frac{\pi n}{2}\right)\exp(-\eta_{sq}n)\mathbf{J}_i-\cos^2\left(\frac{\pi n}{2}\right)\sum_{s'=1}^{N}\exp(-\eta_{s'r}n)S_{s'rsq}(n)\sum_{k=1}^{M}\mathbf{J}_k b_{ki}^{s'r}$$
$$(i=M+1,M+2,\ldots,2M),\tag{55}$$

$$\mathbf{Y}_i^{sq}(n)-\sum_{s'=1}^{N}\sum_{j=M+1}^{2M}\mathbf{Y}_j^{s'q}(n)\sum_{s''=1}^{N}\sum_{k=1}^{M}S_{s'qs''r}(n)C_{s''rsq}(n)b_{jk}^{s'q}b_{ki}^{s''r}$$
$$=\cos^2\left(\frac{\pi n}{2}\right)\exp(-\eta_{sq}n)\mathbf{J}_i-\sin^2\left(\frac{\pi n}{2}\right)\sum_{s'=1}^{N}\exp(-\eta_{s'r}n)C_{s'rsq}(n)\sum_{k=1}^{M}\mathbf{J}_k b_{ki}^{s'r}$$
$$(i=M+1,M+2,\ldots,2M),\tag{56}$$

and

$$\mathbf{X}_i^{sr}(n)-\sum_{s'=1}^{N}\sum_{j=1}^{M}\mathbf{X}_j^{s'r}(n)\sum_{s''=1}^{N}\sum_{k=M+1}^{2M}C_{s'rs''q}(n)S_{s''qsr}(n)b_{jk}^{s'r}b_{ki}^{s''q}$$
$$=\sin^2\left(\frac{\pi n}{2}\right)\exp(-\eta_{sr}n)\mathbf{J}_i-\cos^2\left(\frac{\pi n}{2}\right)\sum_{s'=1}^{N}\exp(-\eta_{s'q})S_{s'qsr}(n)\sum_{k=M+1}^{2M}\mathbf{J}_k b_{ki}^{s'q}$$
$$(i=1,2,\ldots,M),\tag{57}$$

$$\mathbf{Y}_i^{sr}(n)-\sum_{s'=1}^{N}\sum_{j=1}^{M}\mathbf{Y}_j^{s'r}(n)\sum_{s''=1}^{N}\sum_{k=M+1}^{2M}S_{s'rs''q}(n)C_{s''qsr}(n)b_{jk}^{s'r}b_{ki}^{s''q}$$
$$=\cos^2\left(\frac{\pi n}{2}\right)\exp(-\eta_{sr}n)\mathbf{J}_i-\sin^2\left(\frac{\pi n}{2}\right)\sum_{s'=1}^{N}\exp(-\eta_{s'q}n)C_{s'qsr}(n)\sum_{k=M+1}^{2M}\mathbf{J}_k b_{ki}^{s'q}$$
$$(i=1,2,\ldots,M),\tag{58}$$

respectively. Here we have used the notations

$$\eta_{sq}=\tfrac{1}{2}(\mu_s+ip_s), \tag{59}$$

$$\eta_{sr}=\tfrac{1}{2}(\nu_s-iq_s), \tag{60}$$

$$C_{s'qs''r}(n)\equiv C_{s''rs'q}(n)=2\left[\exp(\eta_{s'q}+\eta_{s''r})\cos^2\left(\frac{\pi n}{2}\right)+\sin^2\left(\frac{\pi n}{2}\right)\right]\frac{\exp[-(\eta_{s'q}+\eta_{s''r})n]}{\sinh(\eta_{s'q}+\eta_{s''r})}, \tag{61}$$

$$S_{s'qs''r}(n)\equiv S_{s''rs'q}(n)=2\left[\exp(\eta_{s'q}+\eta_{s''r})\sin^2\left(\frac{\pi n}{2}\right)+\cos^2\left(\frac{\pi n}{2}\right)\right]\frac{\exp[-(\eta_{s'q}+\eta_{s''r})n]}{\sinh(\eta_{s'q}+\eta_{s''r})}, \tag{62}$$

$$b_{jk}^{sq}=2\frac{\alpha_{jk}[\exp(\eta_{sq})]}{\alpha'_{kk}[\exp(\eta_{sq})]}\exp(-\eta_{sq})$$

$$(j=M+1,M+2,\ldots,2M;\ \ i=1,2,\ldots,M), \tag{63}$$

$$b_{jk}^{sr}=-2\frac{\alpha_{jk}[\exp(-\eta_{sr})]}{\alpha'_{kk}[\exp(-\eta_{sr})]}\exp(\eta_{sr})$$

$$(j=1,2,\ldots,M;\ \ k=M+1,M+2,\ldots,2M). \tag{64}$$

The formulas (53)–(58) supplemented by the relations (29) and (30) between $K_{ij}(n|m)$ and $q_\alpha(n),r_\alpha(n)$ are sufficient to unravel the problem of any multisoliton solution of our nonlinear model (12) and (13). For example, the amplitudes of a one-soliton solution restricted by the physically reasonable conditions $t_{\beta\alpha}=t^*_{\alpha\beta}$ and $r_\alpha(n)=q^*_\alpha(n)$ (the reduced amplitudes) are

$$q_\alpha(n)=\frac{b_\alpha(\tau)\sinh\mu\,\exp[ipn+2i\tau\cosh\mu\cos p]}{\sqrt{\sum_{\beta=1}^{M}b_\beta(\tau)b^*_\beta(\tau)}\cosh[\mu(n-x)-2\tau\sinh\mu\sin p]}, \tag{65}$$

$$r_\alpha(n)=q^*_\alpha(n),\quad \alpha=1,2,\ldots,M. \tag{66}$$

Here μ, p, x, and $b_\alpha(\tau)$ are the constant real and time dependent complex integration parameters, respectively, determined through the scattering data of the auxiliary spectral problem by some one-to-one relations. In particular, the quantities $b_\alpha(\tau)$ should satisfy the following set of ordinary differential equations:

$$\dot{b}_\alpha(\tau)=i\sum_{\beta=1}^{M}t_{\alpha\beta}b_\beta(\tau),\quad \alpha=1,2,\ldots,M. \tag{67}$$

Being one-soliton amplitudes, Eqs. (65) and (66) are applicable to each of the models (12), (13) and (14), (15) on an equal basis.

Let us clarify the meaning of the integration parameters. The coordinate x turns to be the mean longitudinal coordinate of the soliton distribution taken at the initial moment $\tau=0$ due merely to the fact that the identity

$$\frac{\sum_{\alpha=1}^{M}\sum_{n=-\infty}^{\infty}nQ_\alpha(n)R_\alpha(n)}{\sum_{\alpha=1}^{M}\sum_{n=-\infty}^{\infty}Q_\alpha(n)R_\alpha(n)}\equiv x+\frac{2}{\mu}\tau\sinh\mu\sin p \tag{68}$$

is fulfilled when calculated on the one-soliton amplitudes. Further, the quantity $2(\sinh\mu\sin p)/\mu$ gives the soliton longitudinal velocity while the quantity $1/\mu$ determines the typical longitudinal size of the soliton distribution. The left-hand side of identity (68) itself is evidently nothing but the definition of the longitudinal coordinate of the soliton distribution taken at an arbitrary moment τ. Finally, the amplitudes $b_\alpha(\tau)$ $(\alpha=1,2,\ldots,M)$ describe the temporal transverse redistribution of soliton density. Indeed, the fraction of the one-soliton density located on the αth chain in accordance with Eqs. (65),(66) and (16),(17) is found to be

$$\frac{Q_\alpha(n)R_\alpha(n)}{\sum_{\beta=1}^{M}Q_\beta(n)R_\beta(n)}=\frac{b_\alpha(\tau)b^*_\alpha(\tau)}{\sum_{\beta=1}^{M}b_\beta(\tau)b^*_\beta(\tau)}\equiv\frac{b_\alpha(\tau)b^*_\alpha(\tau)}{\sum_{\beta=1}^{M}b_\beta(0)b^*_\beta(0)}, \tag{69}$$

where the last step has been reached with the evolution equations (67) and the Hermiticity of the interchain coupling matrix $[t_{\alpha\beta}]$ combined. We will demonstrate the actual temporal interchain redistribution of excitations for particular cases admitting physical applications.

VI. BEATING AND CIRCULAR TRANSVERSE MODES

Being rather general, the model (12) and (13) permits a number of physically interesting ramifications obtainable by merely imposing appropriate restrictions on the coupling constants $t_{\alpha\beta}$. Thus, we are able to model the nonlinear excitations on a multileg ladder lattice unrolled into a two-dimensional strip or combined into a three-dimensional bunch of tightly bound chains. Moreover, in the latter case we are in a position to apply an external magnetic field parallel to the ladder legs in a way similar to that described by Feynman, Leighton, and Sands [44].

We proceed by putting $M=2$ and $t_{\alpha\beta}=(1-\delta_{\alpha\beta})t$ with t a real constant and obtain from Eqs. (12) and (13) a model of nonlinear intramolecular excitations on a two-leg ladder lattice closely related to that of the double helix DNA macromolecule. Then solving Eq. (67) we obtain

$$b_\alpha(\tau)=\tfrac{1}{2}\sum_{\beta=1}^{2}[e^{it\tau}+(-1)^{\alpha-\beta}e^{-it\tau}]b_\beta(0),\quad \alpha=1,2, \tag{70}$$

and consequently

$$\frac{b_\alpha(\tau)b_\alpha^*(\tau)}{\sum_{\beta=1}^{2} b_\beta(\tau)b_\beta^*(\tau)}=\frac{1}{2}-\frac{(-1)^\alpha}{2}[\cos 2\varphi\cos(2t\tau)+\sin(\delta_1-\delta_2)\sin 2\varphi\sin(2t\tau)],\quad \alpha=1,2, \tag{71}$$

where the parametrization $b_1(0)=\exp(i\delta_1)\cos\varphi$, $b_2(0)=\exp(i\delta_2)\sin\varphi$ has been adopted. From Eq. (71) it is evident that there is an interchain beating mode redistributing the excitations between the chains. The beating amplitude is equal to $\sqrt{\cos^2 2\varphi+\sin^2(\delta_1-\delta_2)\sin^2 2\varphi}$ and it can be regulated from zero to unity by means of the parameters δ_1, δ_2, and φ of the initial transverse distribution. Conversely, the beating frequency t/π has a fundamental physical origin and it is determined exclusively by the interchain linear coupling constant t, regardless of any particular solution. Moreover, the effect of interchain beating will be observable only in systems with interchain linear coupling and it is impossible, in principle, e.g., in those of Manakov type [8,18,19]. Thus we can readily reveal a similar effect in the model (12) and (13) on a three-leg ladder lattice ($M=3$) unrolled into a two-dimensional strip $[t_{\alpha\beta}=\varepsilon\delta_{\alpha2}\delta_{2\beta}+(1-\delta_{\alpha\beta}-\delta_{\alpha1}\delta_{3\beta}-\delta_{\alpha3}\delta_{1\beta})t]$, even despite the nonzero difference ε between the energy of intramolecular excitations of the middle and side chains.

Now let us consider the case when $M=3$ and $t_{\alpha\beta}=t\exp(-i\phi/3)\Delta(\alpha-\beta+1)+t\exp(i\phi/3)\Delta(\alpha-\beta-1)$. Here $\Delta(\eta)$ is equal to 1 if the number η is equal to $0,\pm3,\pm6,\dots$ and zero otherwise. Thus at $\phi=0$ the model (12) and (13) describes the chargeless nonlinear intramolecular excitations (or charged ones but without external magnetic field) on a symmetrically rolled three-leg ladder lattice. This model is closely related to the model established for amid-I excitations on α-helix protein macromolecules [2–4]. When the quantity ϕ is nonzero it can be identified with the dimensionless magnetic flux

$$\phi=\frac{e}{c\hbar}|\mathbf{B}|S \tag{72}$$

through the triangular cross section of the symmetrically rolled three-leg ladder lattice, provided the excitations are charged. Here S is the area of a triangular element with vertices situated on molecules of the same unit cell. The constant magnetic field $\mathbf{B}$ is supposed to be directed along the positive direction of the discrete longitudinal coordinate n. It is worth noticing that the magnetic field $\mathbf{B}$ changes the phases of the interchain coupling parameters $t_{\alpha\beta}$, but fortunately in such a way that the nonlinear model of interest does not lose its integrability. Then solving the evolution equations (67) gives rise to

$$b_\alpha(\tau)=\tfrac{1}{3}\exp[2it\tau\cos(\phi/3)]\sum_{\beta=1}^{3}b_\beta(0)+\tfrac{1}{3}\exp[2it\tau\cos(\phi/3-2\pi/3)]\times\sum_{\beta=1}^{3}b_\beta(0)e^{2\pi i(\alpha-\beta)/3}+\tfrac{1}{3}\exp[2it\tau\cos(\phi/3+2\pi/3)]\times\sum_{\beta=1}^{3}b_\beta(0)e^{-2\pi i(\alpha-\beta)/3},\quad \alpha=1,2,3. \tag{73}$$

In general, the corresponding expression for the fraction of one-soliton density located on the αth chain [Eq. (69)] looks rather cumbersome. So we can restrict ourselves to the case when the whole initial soliton density is concentrated on the third chain, $b_\alpha(0)=\delta_{\alpha3}\exp(i\delta_3)$. Then Eq. (69) can be written

$$\frac{b_\alpha(\tau)b_\alpha^*(\tau)}{\sum_{\beta=1}^{3} b_\beta(\tau)b_\beta^*(\tau)}=\tfrac{1}{3}+\tfrac{2}{9}\cos[2\sqrt{3}t\tau\sin(\phi/3)-2\pi\alpha/3]+\tfrac{2}{9}\cos[2\sqrt{3}t\tau\sin(\pi/3-\phi/3)-2\pi\alpha/3]+\tfrac{2}{9}\cos[2\sqrt{3}t\tau\sin(\pi/3+\phi/3)+2\pi\alpha/3],\quad \alpha=1,2,3. \tag{74}$$

According to this formula, the transverse redistribution of soliton density is carried out by three circular traveling waves with frequencies regulated by the external magnetic field. In general, all three modes are different and even incommensurate ones, though at certain particular values of the magnetic field the effects of two-mode degeneracy or two-mode degeneracy accompanied by vanishing of the third mode can be observed.

For example, assuming the magnetic flux to be $\phi=\pm3\pi/2\pm3\pi k$ ($k=0,1,2,\dots$) we see that the last two terms in Eq. (74) become equal. As a result the fraction of one-soliton density located on the αth chain is supported by two different traveling waves on some constant background:

$$\frac{b_\alpha(\tau)b_\alpha^*(\tau)}{\sum_{\beta=1}^{3} b_\beta(\tau)b_\beta^*(\tau)}=\tfrac{1}{3}+\tfrac{2}{9}\cos[\pm(-1)^k2\sqrt{3}t\tau-2\pi\alpha/3]+\tfrac{4}{9}\cos[\pm(-1)^k\sqrt{3}t\tau+2\pi\alpha/3],\quad \alpha=1,2,3. \tag{75}$$

Here the sign plus (+) or minus (−) is chosen depending on whether the electric charge e is positive ($e=+|e|$) or negative ($e=-|e|$).

In another particular case, when $\phi=\pm3\pi k$ ($k=0,1,2,\dots$) one of the frequencies is softened to zero, while the other two coincide, giving rise to a standing mode. As a result the expression (74) for the fraction of one-soliton density is transformed into one standing wave on some constant background:

$$\frac{b_\alpha(\tau)b_\alpha^*(\tau)}{\sum_{\beta=1}^{3} b_\beta(\tau)b_\beta^*(\tau)} = \frac{1}{3} + \frac{2}{9}\cos(2\pi\alpha/3) + \frac{4}{3}\cos(2\pi\alpha/3)\cos(3t\tau), \quad \alpha = 1,2,3. \tag{76}$$

VII. CONCLUSION

In conclusion, we have developed an exactly integrable nonlinear model on a multileg ladder lattice closely related to a wide range of physically important phenomena from nonlinear transport in low-dimensional biological, polymeric, and condensed matter systems to electric pulse propagation in nonlinear transmission lines and light pulse propagation in tunnel- and nonlinearly coupled arrays of optical fibers. In doing this, we have suggested a systematic analytical approach suitable for the needs of nonlinear physics in more than one spatial dimension and have studied the structure of the simplest nonlinear excitations on two- and three-leg ladder lattices.

In particular, we have studied the transverse and longitudinal dynamics of nonlinear excitations on a two-leg ladder lattice and have shown the existence of a transverse beating mode periodically redistributing the soliton density between the chains. Depending on the initial conditions, the relative amplitude of beating can be varied from zero to unity. On the other hand, the frequency of beating has a fundamental physical origin and it is determined by the value of the interchain coupling constant.

In the case of charged nonlinear excitations on a bunchlike three-leg ladder lattice we have managed to describe exactly the effect of a constant magnetic field directed along the chains. Thus it has been shown that the magnetic field breaks the symmetry of the soliton dynamics with respect to clockwise and counterclockwise propagation across the chains, and gives rise to three different circular traveling modes redistributing the soliton density in the transverse direction. Moreover, at some fixed values of the magnetic field these modes can be transformed into two circular traveling modes or even into one standing mode.

In this respect we are inclined to treat each fraction of soliton density $Q_\alpha(n)R_\alpha(n)/\Sigma_{\beta=1}^{M}Q_\beta(n)R_\beta(n)$ as possessing several breathing transverse modes caused by the interchain linear coupling, provided the purely mathematical definition of breathers [45,46] is applied to $Q_\alpha(n)R_\alpha(n)$ [or more generally to $Q_\alpha(n)R_\beta(m)$] instead of to $Q_\alpha(n)$ or $R_\alpha(n)$ separately. Indeed, manipulating the ‘‘physically meaningless’’ quantities $Q_\alpha(n)$ or $R_\alpha(n)$ (in A. S. Davydov’s terminology) we can wrongly include even purely translational soliton modes into the breathing ones. The most direct way to separate spatially constricted translational modes from the breathing ones is to trace the dynamics of the total excitation density $\Sigma_{\alpha=1}^{M}Q_\alpha(n)R_\alpha(n)$ along the chains, which, for example, in the case of the one-soliton solution (65) and (66) related to any multichain integrable model of our type (12) and (13) is nothing but the free movement of a pulselike traveling wave in the longitudinal direction.

In some problems, and in particular those dealing with nonlinear optics, the discretization of amplitudes $q_\alpha(n)$ and $r_\alpha(n)$ with respect to n becomes unnecessary. Then it is reasonable to replace the discrete nonlinear model (12),(13) by its partially continuous equivalent [24], which also happens to be integrable. Nevertheless, the general features of the transverse dynamics of such a partially continuous model should coincide with those of the completely discrete one, inasmuch as the terms responsible for the interchain linear coupling are the same in both models.

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